

Higher Order Cut Elements for the Wave Equation

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Abstract

The scalar wave equation is solved using higher order immersed finite elements. We demonstrate that higher order convergence can be obtained. Small cuts with the background mesh are stabilized by adding penalty terms to the weak formulation. This ensures that the condition numbers of the mass and stiffness matrix are independent of how the boundary cuts the mesh. The penalties consist of jumps in higher order derivatives integrated over the interior faces of the elements cut by the boundary. The dependence on the polynomial degree of the condition number of the stabilized mass matrix is estimated. We conclude that the condition number grows extremely fast when increasing the polynomial degree of the finite element space.

Keywords: Cut Elements, Stabilization, Fictitious, Immersed, XFEM

1. Introduction

Cut elements [1] is an immersed finite element method. For a domain immersed in a background mesh, one solves for the degrees of freedom of the smallest set of elements covering the domain. The inner products in the weak form are taken over the immersed domain. That is, on each element one integrates over the part of the element that is inside the domain. As a result of this, some elements will have a very small intersection with the immersed domain. This will make some eigenvalues of the discrete system very small and in turn, result in poorly conditioned matrices. A suggested way to remedy this is by adding stabilizing terms to the weak formulation. A jump-stabilization was suggested by Burman and Hansbo [2] for the case of piecewise linear elements, where the jump in the normal derivative is integrated over the faces of the elements intersected by the boundary. This form of stabilization has been used with good results in several recent papers, see for example [1, 3, 4, 5], and has also been used for PDEs posed on surfaces in [6, 7].

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Thus, a lot of attention has been directed to the use of lower order elements. Higher order cut elements have received less attention so far. These are interesting in wave propagation problems since for this type of problems the amount of work per dispersion error typically increases slower for higher order methods. Massing et al. [4] suggested stabilizing higher order elements by integrating also jumps in higher derivatives over the faces. This is the intuitive higher order generalization of the stabilization first suggested in [2] and was also mentioned as a possibility by Burman [8].

In this paper, we consider solving the scalar wave equation using higher order cut elements. Both the mass and stiffness matrix are stabilized using the higher order jump-stabilization. We present numerical results showing that the method results in higher order convergence. The time-step restriction of the resulting system is computed numerically and is concluded to be of the same size as for standard finite elements with aligned boundaries. Furthermore, we estimate how the condition number of the stabilized mass matrix depend on the polynomial degree of the basis functions. The estimate suggests that the condition number grows extremely fast with respect to the polynomial degree, which is supported by the numerical experiments. As a remedy for this behavior, we consider lowering the order of the elements close to the boundary. This results in a better condition number, but also in at least half an order lower convergence compared to having elements of full order everywhere. All numerical experiments are performed in 2D, but the generalization to three dimensions is immediate.

One reason why the considered stabilization is attractive is because it is quite easy to implement. Integrals over internal faces occur also in discontinuous Galerkin methods, thus making the implementation similar to what is already supported in many existing libraries.

The suggested jump-stabilization is one but not the only possibility for stabilizing an immersed method. Several papers have used preconditioners to try to overcome problems with ill-conditioning, such as [9, 10, 11]. In [12] a higher order discontinuous Galerkin method was suggested, where the problem of ill-conditioning was solved by merging some of the intersected elements with neighboring elements. One approach which was suggested in [8] is to stabilize the method by making L_2 -projections onto local patches consisting of a small number of elements. Another possibility is to use additional streamline diffusion stabilization on the elements intersected by the boundary, as was done in [13].

This paper is organized in the following way. Notation and some basic problem setup is explained in Section 2.1, the stabilized weak formulation is described in Section 2.2, and the stability of the method is discussed in Section 2.3. Analysis of how fast the condition number increases when increasing the polynomial degree is presented in Section 3, and numerical experiments is presented in Section 4.

2. Theory

2.1. Notation and Setting

Consider the wave equation

$$\begin{aligned}
\ddot{u} &= \nabla^2 u + f(x, t) & x \in \Omega, \quad t \in [0, t_f], \\
u &= g_D(x, t) & x \in \Gamma_D, \quad t \in [0, t_f], \\
\frac{\partial u}{\partial n} &= g_N(x, t) & x \in \Gamma_N, \quad t \in [0, t_f], \\
u &= u_0(x) & x \in \Omega, \quad t = 0, \\
\dot{u} &= v_0(x) & x \in \Omega, \quad t = 0,
\end{aligned} \tag{2.1}$$

posed on a given domain Ω , with $\Gamma_D \cup \Gamma_N = \partial\Omega$. Let $\Omega \subset \mathbb{R}^d$ be immersed in a triangulation, \mathcal{T} , as in Figure 2.1. We assume that each element $T \in \mathcal{T}$ has some part which is inside Ω , that is: $T \cap \Omega \neq \emptyset$. Furthermore, let $\Omega_{\mathcal{T}}$ be the domain that corresponds to \mathcal{T} , that is

$$\Omega_{\mathcal{T}} = \bigcup_{T \in \mathcal{T}} T.$$

Let \mathcal{T}_{Γ} denote the set of elements that are intersected by $\partial\Omega$:

$$\mathcal{T}_{\Gamma} = \{T \in \mathcal{T} : T \cap \partial\Omega \neq \emptyset\},$$

as in Figure 2.2. Let \mathcal{F}_{Γ} denote the faces seen in Figure 2.3. That is, the faces of the elements in \mathcal{T}_{Γ} , excluding the faces that make up $\partial\Omega_{\mathcal{T}}$. To be precise, \mathcal{F}_{Γ} is defined as

$$\mathcal{F}_{\Gamma} = \{F = T_1 \cap T_2 : T_1 \in \mathcal{T}_{\Gamma} \text{ or } T_2 \in \mathcal{T}_{\Gamma}, \quad T_1, T_2 \in \mathcal{T}\}.$$

We assume that our background mesh is sufficiently fine, so that the immersed geometry is well resolved by the mesh. Furthermore, we shall restrict ourselves to meshes as the one in Figure 2.1, where we have a mesh consisting of hypercubes and our coordinate axes are aligned with the mesh faces. That is, the face normals have a nonzero component only in one of the coordinate directions. Denote the distance between two grid points by h .

Consider the situation in Figure 2.4, where two neighboring elements, T_1 and T_2 , are sharing a common face F . Denote by $\partial_n^k v$ the k th directional derivative in the direction of the face normal. That is, fix $j \in \{1, \dots, d\}$ and let the normal of the face, n , be such that

$$n_i = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases} \tag{2.2}$$

then define

$$\partial_n^k v = \frac{\partial^k v}{\partial x_j^k}.$$

In the following we denote by $(\cdot, \cdot)_X$ and $\langle \cdot, \cdot \rangle_Y$ the L_2 scalar products taken over the d and $d - 1$ dimensional domains $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d-1}$. Let $\|\cdot\|_Z$ denote the corresponding norm, and let $|\cdot|_{H^s(Z)}$ denote the H^s -semi-norm. By $[v]$ we shall denote a jump over a face, F , that is: $[v] = v|_{F_+} - v|_{F_-}$.

We shall assume that our basis functions are tensor products of one-dimensional polynomials of order p . In particular, we shall use Lagrange elements with Gauss-Lobatto nodes, in the following referred to as Q_p -elements, $p \in \{1, 2, \dots\}$. Let V_h^p denote a continuous finite element space, consisting of Q_p -elements on the mesh \mathcal{T} :

$$V_h^p = \{v \in C^0(\Omega_{\mathcal{T}}) : v|_T \in Q_p(T)\}. \quad (2.3)$$

Define also the following semi-norm

$$|v|_{\star}^2 = \|\nabla v\|_{\Omega_{\mathcal{T}}}^2 + \frac{1}{h} \|v\|_{\Gamma_D}^2,$$

which is a norm on V_h^p in the case that $\Gamma_D \neq \emptyset$.

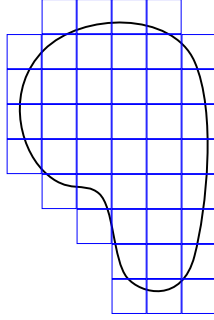


Figure 2.1: Ω immersed in a mesh \mathcal{T} , covering $\Omega_{\mathcal{T}}$.

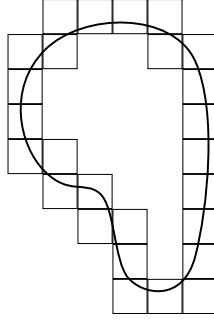


Figure 2.2: Intersected elements \mathcal{T}_{Γ} .

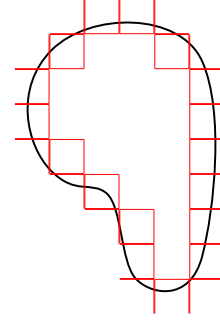


Figure 2.3: Faces \mathcal{F}_{Γ} .

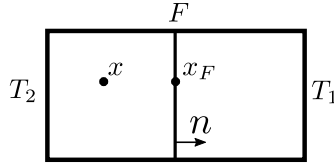


Figure 2.4: Two elements sharing a common face.

2.2. The Stabilized Weak Formulation

Multiplying (2.1) by a test-function, integrating by parts, and applying boundary conditions by Nitsche's method [14] leads to a weak formulation of the following form: find u_h such that for each fix $t \in (0, t_f]$, $u_h \in V_h^p$ and

$$(\ddot{u}_h, v)_{\Omega} + a(u_h, v) = L(v), \quad \forall v \in V_h^p, \quad (2.4)$$

where

$$a(u_h, v) = (\nabla u_h, \nabla v)_\Omega - \left\langle \frac{\partial u_h}{\partial n}, v \right\rangle_{\Gamma_D} - \left\langle u_h, \frac{\partial v}{\partial n} \right\rangle_{\Gamma_D} + \frac{\gamma_D}{h} \langle u_h, v \rangle_{\Gamma_D},$$

$$L(v) = (f, v)_\Omega + \left\langle g_D, \frac{\gamma_D}{h} v - \frac{\partial v}{\partial n} \right\rangle_{\Gamma_D} + \langle g_N, v \rangle_{\Gamma_N}.$$

What makes this different from standard finite elements is that the integration on each element needs to be adapted to the part of the element that is inside the domain. As illustrated in Figure 2.5, some elements will have a very small intersection with the domain. Consider the mass-matrix in (2.4):

$$\tilde{\mathcal{M}}_{ij} = (\phi_i, \phi_j)_\Omega.$$

Note that its smallest eigenvalue is smaller than each diagonal entry:

$$\lambda_{\min} = \min_{z \in \mathbb{R}^N : z \neq 0} \frac{z^T \tilde{\mathcal{M}} z}{z^T z} \leq \tilde{\mathcal{M}}_{ii}, \quad i = 1, \dots, N. \quad (2.5)$$

Depending on the size of the cut with the background mesh some diagonal entries can become arbitrarily close to zero. Thus, both the mass and stiffness matrix can now be arbitrarily ill-conditioned depending on how the cut occurs.

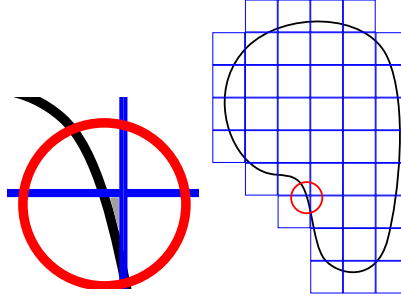


Figure 2.5: An element having a small intersection (in gray) with the domain.

One way to try to remedy this is by adding stabilizing terms, j , to the two bilinear forms

$$\begin{aligned} M(\ddot{u}_h, v) &= (\ddot{u}_h, v)_\Omega + \gamma_M j(\ddot{u}_h, v), \\ A(u_h, v) &= a(u_h, v) + \gamma_A h^{-2} j(u_h, v), \end{aligned} \quad (2.6)$$

where $\gamma_M, \gamma_A > 0$ are penalty parameters. This gives us the following weak formulation: find u_h such that for each fix $t \in (0, t_f]$, $u_h \in V_h^p$ and

$$M(\ddot{u}_h, v) + A(u_h, v) = L(v), \quad \forall v \in V_h^p. \quad (2.7)$$

In [4] a stabilization term of the following form was suggested

$$j(u, v) = \sum_{F \in \mathcal{F}_\Gamma} \sum_{k=1}^p h^{2k+1} \langle [\partial_n^k u], [\partial_n^k v] \rangle_F, \quad (2.8)$$

which in some sense is the intuitive extension of the stabilization which was suggested in [2]. The stabilization in (2.8) was also briefly mentioned as a possibility in [8]. As was discussed in [4] the bilinear form (2.6) can be shown to define a scalar product which is norm equivalent to the L_2 -norm on the whole background mesh:

$$C_L \|v\|_{\Omega_\mathcal{T}}^2 \leq M(v, v) \leq C_U \|v\|_{\Omega_\mathcal{T}}^2, \quad \forall v \in V_h^p, \quad (2.9)$$

and a corresponding equivalence also holds for the gradient

$$\tilde{C}_L \|\nabla v\|_{\Omega_\mathcal{T}}^2 \leq \|\nabla v\|_{\Omega}^2 + \gamma_A h^{-2} j(v, v) \leq \tilde{C}_U \|\nabla v\|_{\Omega_\mathcal{T}}^2, \quad \forall v \in V_h^p. \quad (2.10)$$

The constants C_L and C_U in (2.9) depend on the polynomial degree of our basis functions, but not on how the boundary cuts through the mesh. Let \mathcal{M} denote the mass matrix with respect to the bilinear form M , and $\mathcal{M}_\mathcal{T}$ with respect to the scalar product on the background mesh, that is:

$$\begin{aligned} \mathcal{M}_{ij} &= M(\phi_i, \phi_j), \\ (\mathcal{M}_\mathcal{T})_{ij} &= (\phi_i, \phi_j)_{\Omega_\mathcal{T}}. \end{aligned}$$

Now, (2.9) implies that the condition number, $\kappa(\mathcal{M})$, of \mathcal{M} is bounded by the condition number of $\mathcal{M}_\mathcal{T}$:

$$\kappa(\mathcal{M}) \leq \frac{C_U}{C_L} \kappa(\mathcal{M}_\mathcal{T}). \quad (2.11)$$

Starting from (2.10) one can also show that $A(\cdot, \cdot)$ is coercive in V_h^p with respect to the $|\cdot|_\star$ -semi-norm on the background mesh:

$$\exists C_c > 0: \quad C_c |v|_\star^2 \leq A(v, v), \quad \forall v \in V_h^p. \quad (2.12)$$

This follows by the same procedure as in [4], given the following inverse inequality

$$h^{1/2} \left\| \frac{\partial v}{\partial n} \right\|_{\Gamma \cap T} \leq C p^2 \|\nabla v\|_T, \quad \forall v \in V_h^p, \quad (2.13)$$

which we derive in Lemma 7 in Appendix A.¹

The stabilization in (2.8) is the basic form of stabilization that we shall consider. However, each time we differentiate we will introduce some dependence on the polynomial degree. It therefore seems reasonable that each term in the

¹This was also shown for piecewise linear basis functions in the proof of Lemma 4 in [15].

sum should be scaled in some way. Because of this, we consider a stabilization of the following form:

$$j(u, v) = \sum_{F \in \mathcal{F}_\Gamma} \sum_{k=1}^p w_k \frac{h^{2k+1}}{(2k+1)(k!)^2} \langle [\partial_n^k u], [\partial_n^k v] \rangle_F, \quad (2.14)$$

where $w_j \in \mathbb{R}^+$ are some weights, which we are free to choose as we wish. The choice of weights will determine how large our constants C_U , C_L in (2.11) are and in turn influence how well conditioned our system is.

2.3. Stability

The bilinear forms in (2.7) are symmetric, which is a quite important property, since it in the end will guarantee stability of the system. In order to show stability we want a bound on $\|u\|_{\Omega_\tau}$. Define an energy, E , of the form

$$E(t) := \frac{1}{2} (M(\dot{u}_h, \dot{u}_h) + A(u_h, u_h)). \quad (2.15)$$

Since both bilinear forms are at least positive semi-definite, this energy has the property $E \geq 0$. The symmetry now allows us to show that for a homogeneous system,

$$f(x) = 0, \quad g_D(x) = 0, \quad g_N(x) = 0,$$

the energy is conserved:

$$\frac{dE}{dt} = M(\ddot{u}_h, \dot{u}_h) + A(u_h, \dot{u}_h) \stackrel{(2.7)}{=} 0, \quad (2.16)$$

so that

$$E(t) = E(0). \quad (2.17)$$

By the definition of the energy together with (2.9) and (2.12) this immediately implies that $\|\dot{u}_h\|_{\Omega_\tau}$ and $\|\nabla u_h\|_{\Omega_\tau}$ are both bounded. For the case $\Gamma_D \neq \emptyset$ the semi-norm $|\cdot|_*$ is a norm for the space V_h^p and (2.12) implies that $\|u_h\|_{\Omega_\tau}$ is also bounded. When $\Gamma_D = \emptyset$ we can use that

$$2\|u_h\|_{\Omega_\tau} \frac{d}{dt} \|u_h\|_{\Omega_\tau} = \frac{d}{dt} \|u_h\|_{\Omega_\tau}^2 = 2(u_h, \dot{u}_h)_{\Omega_\tau} \leq 2\|u_h\|_{\Omega_\tau} \|\dot{u}_h\|_{\Omega_\tau},$$

which gives us

$$\frac{d}{dt} \|u_h\|_{\Omega_\tau} \leq \|\dot{u}_h\|_{\Omega_\tau}.$$

By integrating we obtain that $\|u_h\|_{\Omega_\tau}$ is bounded since $\|\dot{u}_h\|_{\Omega_\tau}$ is bounded:

$$\|u_h(t)\|_{\Omega_\tau} \leq \|u_h(0)\|_{\Omega_\tau} + \int_0^{t_f} \|\dot{u}_h\|_{\Omega_\tau} dt.$$

Thus the system is stable.

In total the system (2.7) discretizes to a system of the form

$$\mathcal{M} \frac{d^2 \xi}{dt^2} + \mathcal{A} \xi = \mathcal{F}(t), \quad (2.18)$$

with $\mathcal{M}, \mathcal{A} \in \mathbb{R}^{N \times N}$, and $\xi \in \mathbb{R}^N$, and where

$$\mathcal{A}_{ij} = A(\phi_i, \phi_j).$$

When solving this system in time we will have a restriction on the time-step, τ , of the form

$$\tau \leq \alpha C_{FL} h,$$

where α is a constant which depends on the time-stepping algorithm. The constant C_{FL} is determined as

$$C_{FL} = \frac{h^{-1}}{\sqrt{\lambda_{\max}}}, \quad (2.19)$$

where λ_{\max} is the largest eigenvalue of the generalized eigenvalue problem: find (x, λ) such that

$$\mathcal{A}x - \lambda \mathcal{M}x = 0, \quad x \in \mathbb{R}^N.$$

Since we would expect that the added stabilization has some effect on the C_{FL} -constant we will investigate it numerically in Section 4. It turns out that it is not worse than for the standard case when the boundaries are aligned with the mesh.

3. Analysis of the Condition Number of the Mass Matrix

We would like to choose the weights in (2.14) in order to minimize the condition number of the mass matrix. This will require us to know how the condition number depends on the weights and the polynomial degree. To determine this, we follow essentially the same path as in [4] and keep track of the weights and the polynomial dependence of the involved inequalities. In the following, we denote by C various constants which do not depend on h and p , unless explicitly stated otherwise. We shall also by w denote the vector $w = (w_1, \dots, w_p)$, where w_j are the weights in the stabilization term (2.14). We can now derive the following inequality, which is a weighted version of Lemma 5.1 in [4].

Lemma 1. *Given two neighboring elements, T_1 and T_2 , sharing a face F (as in Figure 2.4), and $v \in V_h^p$, we have that:*

$$\|v\|_{T_1}^2 \leq L(w) \left(\|v\|_{T_2}^2 + \sum_{k=1}^p w_k \frac{h^{2k+1}}{(2k+1)(k!)^2} \|[\partial_n^k v]\|_F^2 \right), \quad (3.1)$$

where

$$L(w) = C_1(p) + \sum_{k=1}^p \frac{1}{w_k}. \quad (3.2)$$

Proof. Denote by v_i the restriction of v to T_i and then extended by expression to the whole of $T_1 \cup T_2$. For a point $x \in T_1 \cup T_2$ denote by $x_F(x)$ the projection of x onto the face, as in Figure 2.4. We may now Taylor expand from the face:

$$v_i(x) = \sum_{k=0}^p \frac{1}{k!} \partial_n^k v_i(x_F(x)) |x - x_F(x)|^k,$$

and consequently

$$v_1(x) = v_2(x) + \sum_{k=1}^p \frac{1}{k!} [\partial_n^k v(x_F)] |x - x_F|^k. \quad (3.3)$$

Now introduce the following weighted $l^2(\mathbb{R}^{p+1})$ -norm:

$$\|z\|_\alpha^2 := \sum_{k=0}^p \alpha_k z^2,$$

where $\alpha_k > 0$ and $z \in \mathbb{R}^{p+1}$. If $\|\cdot\|_1$ denotes the usual $l^1(\mathbb{R}^{p+1})$ -norm we have that:

$$\|z\|_1^2 \leq C_\alpha \|z\|_\alpha^2, \quad (3.4)$$

where

$$C_\alpha = \sum_{k=0}^p \frac{1}{\alpha_k}. \quad (3.5)$$

Taking the $L_2(T_1)$ -norm of (3.3) and using (3.4) now results in:

$$\|v_1\|_{T_1}^2 \leq C_\alpha \left(\alpha_0 \|v_2\|_{T_1}^2 + \sum_{k=1}^p \alpha_k \frac{h^{2k+1}}{(2k+1)(k!)^2} \|[\partial_n^k v]\|_F^2 \right). \quad (3.6)$$

Since v_2 lies in a finite dimensional polynomial space on $T_1 \cup T_2$ the norms on T_1 and T_2 are equivalent:

$$\|v_2\|_{T_1}^2 \leq C_1 \|v_2\|_{T_2}^2,$$

where $C_1 = C_1(p)$. Using this in (3.6) and choosing

$$\begin{aligned} \alpha_0 &= 1/C_1, \\ \alpha_k &= w_k, \quad k = 1, \dots, p \end{aligned}$$

gives us (3.1). □

Lemma 1 will now allow us to bound the bilinear form M from below.

Lemma 2. *A lower bound for $M(v, v)$ is:*

$$\|v\|_{\Omega_\tau}^2 \leq C_l L(w)^{N_J} M(v, v), \quad (3.7)$$

where N_J is some sufficiently large integer and C_l is a constant independent of h and p .

Proof. Let $T_0 \in \mathcal{T}_\Gamma$ and let $\{T_i\}_{i=1}^{N-1}$ (with $T_i \in \mathcal{T}_\Gamma$) be a sequence of elements that need to be crossed in order to get to an element $T_N \in \mathcal{T} \setminus \mathcal{T}_\Gamma$ as in Figure 3.1, and denote $F_i = T_{i-1} \cap T_i$. By using (3.1) we get

$$\|v\|_{T_0}^2 \leq L(w)^N \left(\|v\|_{T_N}^2 + \sum_{i=1}^N \sum_{k=1}^p w_k \frac{h^{2k+1}}{(2k+1)(k!)^2} \|[\partial_n^k v]\|_{F_i}^2 \right),$$

where we have used that $L(w) \geq 1$ (since at least $C_1 \geq 1$). Let now $N_J \geq 1$ denote some upper bound on the maximum number of jumps that needs to be made in the mesh. If our geometry is well resolved by our background mesh N_J is a small integer. This gives us

$$\|v\|_{\Omega_T}^2 = \sum_{T \in \mathcal{T}_\Gamma} \|v\|_T^2 + \sum_{T \in \mathcal{T} \setminus \mathcal{T}_\Gamma} \|v\|_T^2 \leq CL(w)^{N_J} \left(\sum_{T \in \mathcal{T} \setminus \mathcal{T}_\Gamma} \|v\|_T^2 + j(v, v) \right),$$

from which (3.7) follows. \square

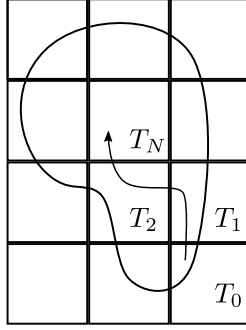


Figure 3.1: A sequence of jumps from a boundary element $T_0 \in \mathcal{T}_\Gamma$ to an inside element T_N .

We proceed by estimating how a bound on the jumps depend on the polynomial degree.

Lemma 3. *For the jumps in the normal derivative we have that:*

$$\|[\partial_n^k v]\|_F^2 \leq C_k \frac{p^{4k+2}}{h^{2k+1}} \left(\|v\|_{T_F^+}^2 + \|v\|_{T_F^-}^2 \right), \quad \text{for } k = 1, 2, \dots, p \quad (3.8)$$

where T_F^+ and T_F^- denotes the two elements sharing the face F .

Proof. Note first that

$$\|[\partial_n^k v]\|_F^2 \leq 2 \left(\|\partial_n^k v_1\|_F^2 + \|\partial_n^k v_2\|_F^2 \right). \quad (3.9)$$

We shall need the following inequalities:

$$\|v\|_F \leq C \frac{p}{\sqrt{h}} \|v\|_T, \quad (3.10)$$

$$|v|_{H^s(T)} \leq C^s \frac{p^{2s}}{h^s} \|v\|_T, \quad (3.11)$$

which were discussed² in [16]. Although (3.11) holds for a whole element we shall use the corresponding inequality applied to a face:

$$|v|_{H^s(F)} \leq C^s \frac{p^{2s}}{h^s} \|v\|_F. \quad (3.12)$$

This is valid since a function v in the tensor product space over T will have a restriction $v|_F$ in the tensor product space over the face F . Note that the constants, C , in (3.11) and (3.12) are not necessarily the same. By combining (3.9), (3.10) and (3.12) we obtain (3.8). \square

Using Lemma 3 we can now bound the bilinear form $M(\cdot, \cdot)$ from above.

Lemma 4. *An upper bound for $M(v, v)$ is:*

$$M(v, v) \leq (1 + C_g G(w)) \|v\|_{\Omega_T}^2, \quad (3.13)$$

where

$$G(w) = \sum_{k=1}^p w_k \frac{p^{4k+2}}{(2k+1)(k!)^2}, \quad (3.14)$$

and C_g is a constant independent of h and p .

Proof. Using the definition of $j(\cdot, \cdot)$ and applying Lemma 3 on each order of derivatives in the sum individually we have

$$j(v, v) \leq C G(w) \sum_{F \in \mathcal{F}_T} \left(\|v\|_{T_F^-}^2 + \|v\|_{T_F^+}^2 \right).$$

Let N_F denote the number of faces that an element has in \mathbb{R}^d . We now have

$$\sum_{F \in \mathcal{F}_T} \left(\|v\|_{T_F^+}^2 + \|v\|_{T_F^-}^2 \right) \leq 2N_F \sum_{T \in \mathcal{T}} \|v\|_{T_F}^2 \leq 2N_F \|v\|_{\Omega_T}^2, \quad (3.15)$$

so we finally obtain:

$$j(v, v) \leq C_g G(w) \|v\|_{\Omega_T}^2,$$

which gives us (3.13). \square

Using Lemma 2 and 4 we now have the following bound on the condition number.

²In particular see (4.6.4) and (4.6.5) in Theorem 4.76, together with the argumentation leading to Corollary 3.94

Lemma 5. *An upper bound for the condition number of the mass matrix is*

$$\kappa(\mathcal{M}) \leq C_M(w)\kappa(\mathcal{M}^*), \quad (3.16)$$

where

$$C_M = C_l L(w)^{N_J} (1 + C_g G(w)) \kappa(\mathcal{M}^*).$$

Proof. Let $\lambda(\cdot)$ denote eigenvalues. From Lemma 2 and 4 we obtain

$$\frac{\lambda_{\min}(\mathcal{M}^*)}{C_l L(w)^{N_J}} \leq \lambda_{\min}(\mathcal{M}),$$

$$\lambda_{\max}(\mathcal{M}) \leq (1 + C_g G(w)) \lambda_{\max}(\mathcal{M}^*),$$

which gives us (3.16). \square

Here, we would like to choose the weights in order to minimize the constant C_M . However, we have the following unsatisfactory result, which shows that no matter how we choose the weights our analysis cannot yield a p -independent bound on the conditioning.

Lemma 6. *The constant $C_M(w)$ in Lemma 5 fulfills $C_M(w) \geq C_0 P(p)$, where C_0 does not depend on p . Here $P(p)$ is the function*

$$P(p) = \sum_{k=1}^p \frac{p^{4k+2}}{(k!)^2 (2k+1)}, \quad (3.17)$$

which is independent of the choice of weights w .

Proof. First note that

$$C_l L(w)^{N_J} (1 + C_g G(w)) \geq C_l C_g G(w) L(w)^{N_J} \geq C_l C_g L(w) G(w).$$

Now we have

$$\begin{aligned} L(w) G(w) &\geq \sum_{k=1}^p \left(w_k \frac{p^{4k+2}}{(2k+1)(k!)^2} \right) \sum_{k=1}^p \left(\frac{1}{w_k} \right) \geq \\ &\sqrt{\sum_{k=1}^p \left(w_k^2 \left(\frac{p^{4k+2}}{(2k+1)(k!)^2} \right)^2 \right)} \sqrt{\sum_{k=1}^p \left(\frac{1}{w_k} \right)^2} \geq P(p), \end{aligned}$$

where we first used that the $l^1(\mathbb{R}^p)$ -norm is greater than the $l^2(\mathbb{R}^p)$ -norm and finally Cauchy-Schwartz. From this the result follows. \square

The function $P(p)$ increases incredibly fast when increasing the polynomial degree. This result could reflect either:

1. The analysis leading to Lemma 5 is not sharp. The bound C_M is too generous, and a better bound exists.

2. The bound in Lemma 5 is not unnecessarily generous, so that the constant C_M is in some sense “tight”. This means that the condition number of the stabilized mass matrix (2.6) will grow faster than the function $P(p)$, regardless of the choice of weights.

Alternative 2 is rather devastating from a time-stepping perspective, since in order to time-step (2.18) an inverse of the mass matrix needs to be available in each time-step. If this inversion is done with an iterative method the number of required iterations until convergence is going to be large.

A combination of these two alternatives is, of course, possible. The estimate in Lemma 5 could be too pessimistic, but even the optimal bound increases incredibly fast. Given the results in Section 4 this appears to be the most plausible alternative.

Lowering the Order at the Boundary

As a remedy to the expected poor behavior of the condition number of the mass matrix, we shall consider lowering the order of the elements close to the boundary. This will be done in the way illustrated in Figure 3.2. This idea is based on two observations:

- In finite difference methods it is possible to lower the order close to the boundary and still get full convergence [17, 18].
- By using lower order elements close to the boundary we only need to stabilize jumps in derivatives up to order $p - 1$.

Let $N_F(T)$ denote the neighboring element of the element T sharing the face F with T . We now construct a new finite element space \tilde{V}_h^p in the following way. Elements which are intersected or have an intersected neighbor are lowered one order compared to the interior of the domain. More precisely:

$$\tilde{V}_h^p = \left\{ v \in C^0(\Omega_{\mathcal{T}}) : \begin{cases} v|_T \in Q_{p-1}(T), & T \in \mathcal{T}_\Gamma \text{ or } \exists F : N_F(T) \in \mathcal{T}_\Gamma \\ v|_T \in Q_p(T), & \text{Otherwise} \end{cases} \right\}. \quad (3.18)$$

This form of space will introduce hanging nodes between elements of different orders. This can be solved in several ways, but in the experiments in Section 4 we treat this by adding constraints that enforce continuity at the hanging nodes.

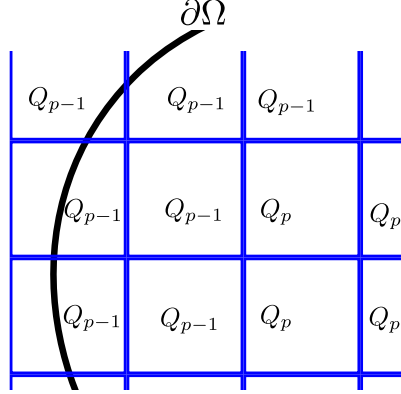


Figure 3.2: Space, \tilde{V}_h^p , constructed by lowering the order of the elements close to the boundary.

3.1. Choosing Weights in the Jump-Stabilization

In order to do a computation, we are forced to make some choice of the weights w_i . The essence of Lemma 6 is that we can bound $L(w)G(w)$ from below. So in order to choose weights let us assume that:

$$\kappa(\mathcal{M}) \propto L(w)G(w).$$

From Lemma 4 it is seen that choosing $w_i \gg 1$ makes $G(w)$ very large. In the same way, Lemma 2 tells us that choosing $w_i \ll 1$ for some i makes $L(w)$ very large. From this observation it seems reasonable to try to enforce both bounds to be of about the same magnitude. In this way, we minimize $L(w)G(w)$ with respect to w and enforce $G(w) = L(w)$. This leaves us with

$$\nabla_w L + \nabla_w G = 0,$$

where ∇_w denotes the gradient with respect to w . This now gives us the following choice of weights:

$$w_k = k! \frac{\sqrt{2k+1}}{p^{2k+1}}. \quad (3.19)$$

There is no reason why this argumentation should lead to the optimal choice of weights, but it seems reasonable that this is *not a particularly bad choice*.

4. Numerical Experiments

In the following, we shall consider a problem both with the finite element space V_h^p from (2.3) and with the finite element spaces \tilde{V}_h^p from (3.18). The weights from (3.19) are used, with p determined by the order of the polynomials at the boundary. In addition, the following parameters are used

$$\begin{aligned} \gamma_M &= 0.25/\sqrt{3}, \\ \gamma_A &= 0.5/\sqrt{3}, \\ \gamma_D &= 2.5p(p+1). \end{aligned}$$

The errors are computed in norms which are some quantities integrated over the domain Ω . It is worth noting that the geometry of Ω is represented by a level set function, ψ_h . Both for the case when $u \in V_h^p$ and when $u \in \tilde{V}_h^p$ the level set function is an element in the space

$$W_h^p = \{v \in C_0(\Omega_{\mathcal{T}_B}) : v|_T = Q_p(T)\},$$

where

$$\Omega_{\mathcal{T}_B} = \bigcup_{T \in \mathcal{T}_B} T,$$

and where \mathcal{T}_B is a larger background mesh from which \mathcal{T} was created. In order to perform the quadratures over the elements intersected by the boundary we have used an algorithm by Saye [19], which generates the quadrature rules on the intersected elements with respect to ψ_h . Thus, also the errors of the solution are calculated with respect to this approximation of the geometry. That is, the L_2 -norms are approximated as

$$\begin{aligned} \|\cdot\|_{\Omega} &\approx \|\cdot\|_{\psi_h < 0}, & \psi_h &\in V_h^p, \\ \|\cdot\|_{\partial\Omega} &\approx \|\cdot\|_{\psi_h = 0}, \end{aligned}$$

where ψ_h is initialized by L_2 -projecting the analytic level set function onto the space W_h^p . Convergence-rates are estimated as

$$\frac{\log(e_i/e_{i+1})}{\log(h_i/h_{i+1})},$$

where e_i denotes an error corresponding to mesh size h_i .

Time-stepping is performed with a classical fourth order explicit Runge-Kutta, after rewriting the system (2.18) as a first order system in time. A time step, τ , of size

$$\tau = \frac{0.9}{p^2} h$$

is used. Implementation was done in the library `deal.II` [20].

4.1. Standard Reference Problem with Aligned Boundary

It is relevant to compare some of the properties of the mass and stiffness matrix with standard (non-immersed) finite elements. For this purpose Table 1 shows the C_{FL} number, the minimal and maximal eigenvalues of the mass matrix, together with the condition number of the mass matrix, for the non-immersed case. The table also shows how these change when changing the order of the elements in the space. The values were computed on a rectangular grid with size $[-1.5, 1.5] \times [-1.5, 1.5]$, with Neumann boundary conditions, for a single grid size $h = 0.036$. As for the immersed case, quadrilateral Lagrange elements with Gauss-Lobatto nodes were used. The C_{FL} -constant was computed according to (2.19). Since all eigenvalues should be proportional to h^2 , the eigenvalues have been scaled by h^{-2} for easier comparison.

Table 1: Quantities computed with (non-immersed) finite elements aligned with the boundary.

p	C_{FL}	$h^{-2}\lambda_{\min}(\tilde{\mathcal{M}})$	$h^{-2}\lambda_{\max}(\tilde{\mathcal{M}})$	$\kappa(\tilde{\mathcal{M}})$
1	0.204	6.25e-02	9.99e-01	16.0
2	0.091	1.15e-02	3.36e-01	29.2
3	0.054	3.67e-03	1.74e-01	47.3

4.2. An Inner Problem

Let Ω be a disk domain, centered at origo, with radius $R = 1$, and enforce homogeneous Dirichlet boundary condition along the boundary

$$u|_{\partial\Omega} = 0.$$

Let J_0 denote the 0:th order Bessel-function and let α_n denote its n :th zero. By starting from initial conditions:

$$\begin{aligned} u|_{t=0} &= J_0(\alpha_n \frac{\|x\|}{R}), \\ \frac{\partial u}{\partial t} \Big|_{t=0} &= 0, \end{aligned}$$

we can calculate the error in our numerical solution with respect to the analytic solution:

$$u(x, t) = J_0(\alpha_n \frac{\|x\|}{R}) \cos(\omega_n t), \quad \omega_n = \frac{\alpha_n}{R}.$$

A few snapshots of the numerical solution are shown in Figure 4.1. The problem was solved with the given method until an end-time, t_f , corresponding to a three periods:

$$t_f = 3T_p, \quad T_p = \frac{2\pi}{\omega_n}.$$

At this end-time the errors were computed.

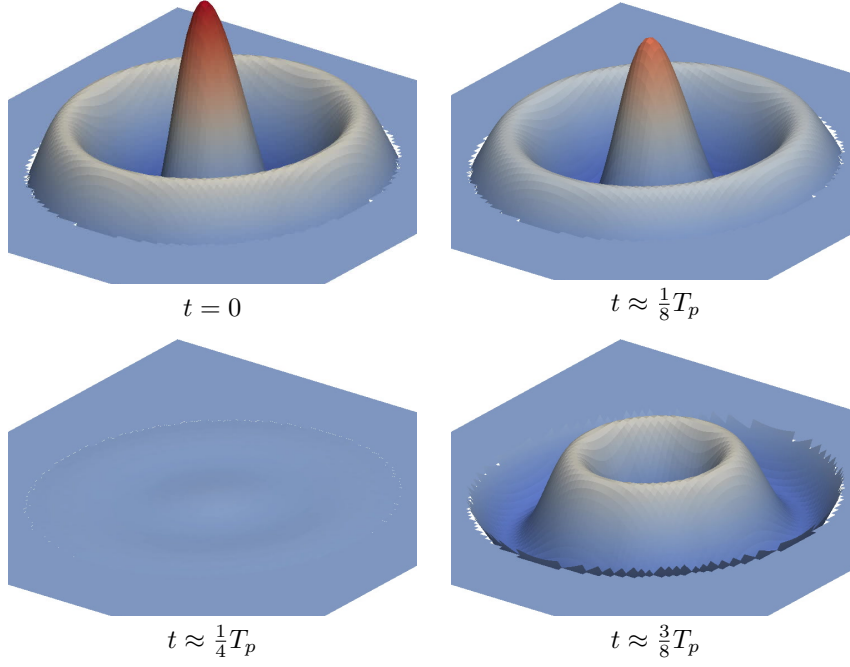


Figure 4.1: Snapshots of the vibrating membrane problem.

Results Using the Space V_h^p

The calculated errors and estimated convergence rates for the different element orders are shown in Tables 2 to 4. One would expect that the $L_2(\Omega)$ - and $L_2(\partial\Omega)$ -errors are proportional to h^{p+1} and that the $H^1(\Omega)$ -error is proportional to h^p . The rates of the $L_2(\partial\Omega)$ -error is slightly low when using Q_2 -elements. Otherwise, the rates are higher than expected.

The computed C_{FL} -constant for the different orders of elements are shown in Table 5. Tabulated are the value after calculating the CFL-number according to (2.19) for each grid-level and then taking the mean over all grid sizes. The C_{FL} -constant is not worse than for the non-immersed case in Table 1.

How the condition number of the mass matrix depend on the grid size is shown in Figure 4.2, for the different orders of p . We see that the condition numbers are essentially constant when refining h , in agreement with (2.9). We also see that the condition numbers increase extremely rapidly when increasing the polynomial degree, as predicted by Lemma 6. In particular when compared to the condition numbers of the non-immersed case in Table 1. The dashed lines in Figure 4.2 denote the function $CP(p)$, where P is the function from (3.17) and C is a constant determined by fitting it to the mean (with respect to h) of the condition numbers for V_h^1 . The estimate from Lemma 6 does not agree particularly well with the numerical results.

The minimal and maximal eigenvalues for the different polynomial orders and refinements are seen in Figure 4.3. As can be seen, the scaled eigenvalues are

essentially constant with respect to h . Thus the dependence on h is as expected. We see that the minimal eigenvalues decrease quite fast when increasing the polynomial degree, and that they are substantially smaller than in the non-immersed case in Table 1. At the same time, the maximal eigenvalues grow and are larger than in Table 1.

The spectra of the mass matrix corresponding to the coarsest refinement level are shown in Figure 4.4. As can be seen, there is a “kink” in the spectrum. This appears at the eigenvalue λ_i , corresponding to $i/n \approx 0.45$ (n being the number of degrees of freedom for this grid size) for V_h^3 . The same behavior can be seen in the spectrum for V_h^2 , but at the eigenvalue corresponding to $i/n \approx 0.8$.

Table 2: Errors when using the space V_h^1 .

h	$L_2(\Omega)$		$H^1(\Omega)$		$L_2(\partial\Omega)$	
1.000e-01	7.017e-01	-	6.108e+00	-	1.383e-02	-
5.000e-02	2.288e-01	1.62	2.012e+00	1.60	4.641e-03	1.58
2.500e-02	2.503e-02	3.19	3.007e-01	2.74	7.876e-04	2.56
1.250e-02	4.641e-03	2.43	1.172e-01	1.36	8.217e-05	3.26

Table 3: Errors when using the space V_h^2 .

h	$L_2(\Omega)$		$H^1(\Omega)$		$L_2(\partial\Omega)$	
1.000e-01	1.107e-02	-	2.036e-01	-	5.871e-03	-
5.000e-02	1.302e-03	3.09	3.944e-02	2.37	9.376e-04	2.65
2.500e-02	1.029e-04	3.66	9.699e-03	2.02	1.480e-04	2.66

Table 4: Errors when using the space V_h^3 .

h	$L_2(\Omega)$		$H^1(\Omega)$		$L_2(\partial\Omega)$	
1.000e-01	6.104e-04	-	2.534e-02	-	4.403e-04	-
5.000e-02	2.153e-05	4.83	2.130e-03	3.57	1.725e-05	4.67
2.500e-02	4.346e-07	5.63	1.275e-04	4.06	5.547e-07	4.96

Table 5: Computed CFL-numbers.

	V_h^1	V_h^2	V_h^3
C_{FL}	0.37	0.13	0.07

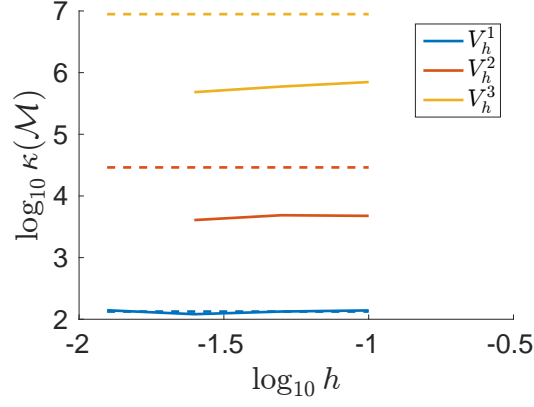


Figure 4.2: Condition number of the mass matrix when using the space V_h^p , for different h and p . The dashed lines denotes estimates according to the function $P(p)$.

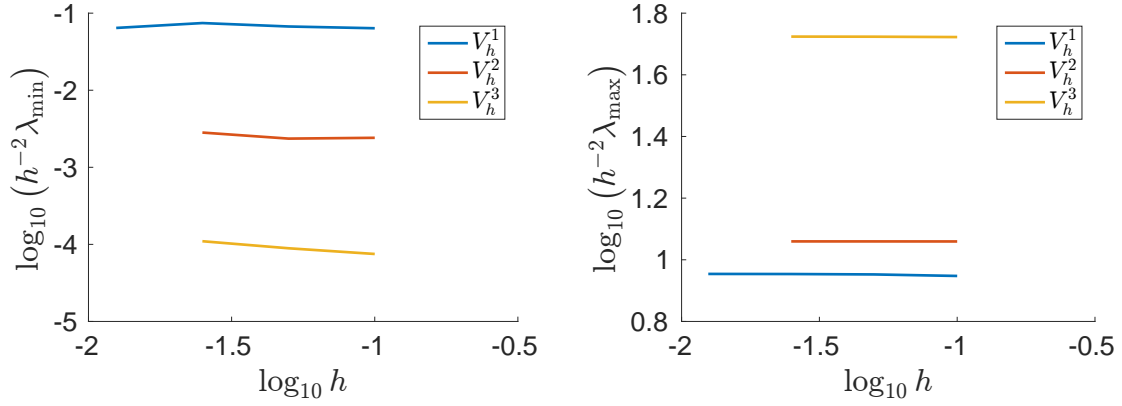


Figure 4.3: Maximal/minimal eigenvalues (scaled by h^{-2}) of the mass matrix, when using the space V_h^p , for different h and p .

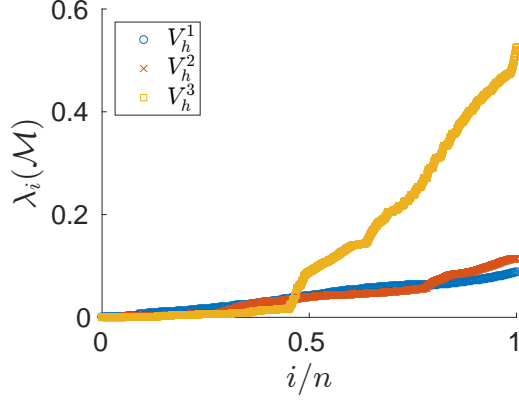


Figure 4.4: Spectra of the mass matrix when using the spaces V_h^1 - V_h^3 , for $h = 0.1$.

Results Using the Space \tilde{V}_h^p

The errors and convergence rates when using the space \tilde{V}_h^2 are shown in Table 6. For the errors in $L_2(\Omega)$ - and $H^1(\Omega)$ -norm it seems that we obtain the rate corresponding to the highest element (Q_2) in the space. Not unexpectedly we seem to get the lower order convergence for the $L_2(\Omega)$ -error along the boundary. However, when using the space \tilde{V}_h^3 the situation appears to be different. Here, it seems that one loses at least half an order for the rates of the $L_2(\Omega)$ - and $H^1(\Omega)$ -errors.

How the condition number of the mass-matrix depends on h for the two spaces \tilde{V}_h^2 and \tilde{V}_h^3 are shown in Figure 4.5. By comparing to Figure 4.2 we see that the condition number of the space V_h^p is essentially the same as for the corresponding space with the lowest order element everywhere. That is:

$$\kappa(\mathcal{M}_{\tilde{V}_h^p}) \approx \kappa(\mathcal{M}_{V_h^{p-1}}),$$

which is not surprising since we expect that all ill-conditioning is due to the added penalty term, $j(\cdot, \cdot)$. The minimal and maximal eigenvalues of the mass matrix and the CFL-numbers for the space \tilde{V}_h^p look essentially the same as for the space V_h^{p-1} . The mass matrix spectra for these spaces are shown in Figure 4.6, for the coarsest grid level. The spectra look similar to the spaces V_h^{p-1} .

Table 6: Errors when using the space \tilde{V}_h^2 .

h	$L_2(\Omega)$		$H^1(\Omega)$		$L_2(\partial\Omega)$	
1.000e-01	2.528e-02	-	5.101e-01	-	6.211e-03	-
5.000e-02	5.526e-03	2.19	1.670e-01	1.61	1.467e-03	2.08
2.500e-02	4.055e-04	3.77	2.274e-02	2.88	1.953e-04	2.91
1.250e-02	4.156e-05	3.29	4.086e-03	2.48	4.144e-05	2.24

Table 7: Errors when using the space \tilde{V}_h^3 .

h	$L_2(\Omega)$		$H^1(\Omega)$		$L_2(\partial\Omega)$	
1.000e-01	3.290e-03	-	1.283e-01	-	2.935e-03	-
5.000e-02	3.925e-04	3.07	2.570e-02	2.32	4.460e-04	2.72
2.500e-02	3.547e-05	3.47	4.766e-03	2.43	4.705e-05	3.24

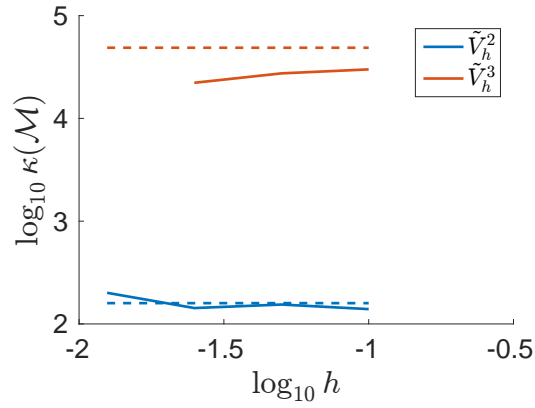


Figure 4.5: Condition number of the mass matrix when using the space \tilde{V}_h^p , for different h and p . The dashed lines denotes estimates according to the function $P(p)$.

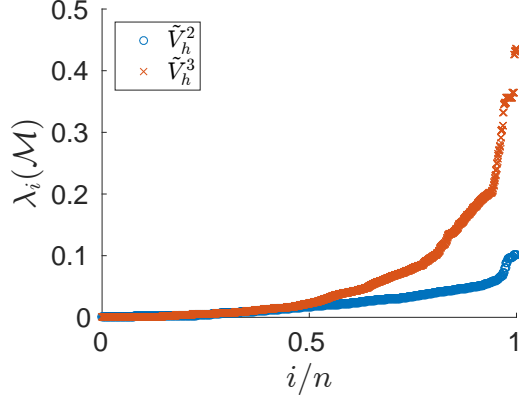


Figure 4.6: Spectra of the mass matrix when using the spaces \tilde{V}_h^2 and \tilde{V}_h^3 , for $h = 0.1$.

5. Discussion

The results in Section 4.2 show that it is possible to solve the problem and obtain up to 4th order convergence. In particular, it is also promising that the CFL-condition is not stricter than for the non-immersed case. However, both the theoretical results in Lemma 6 and the results in Section 4.2 show that there are problems with the conditioning of the mass matrix. It should be emphasized that even if the added stabilization creates some new problems it is by far better than using no stabilization at all. With the added stabilization the method can be proved to be stable, which is essential.

It would, of course, be advantageous if one would be able to create a stabilization which does not lead to conditioning problems. However, the prospects for creating a good preconditioner for the mass matrix is rather good, since the stabilization maintains the symmetry of the mass matrix and since one obtains bounds on its spectrum from the analysis.

The choice of the weights in (3.19) were based on hand-waving arguments and can, therefore, be criticized. We have tried other choices of weights but have not presented the results here. This is mainly because they give similar results and we have no reason to believe that there exists a choice which makes the condition number significantly better.

The idea of lowering the order of the elements close to the boundary worked quite well for the space \tilde{V}_h^2 . We obtained the convergence corresponding to the higher elements in the space, but the condition number corresponding to the lower order elements. Sadly this was not the case when increasing the element order further and going to the space \tilde{V}_h^3 . Thus, the procedure does not appear to be a plausible solution for going to higher orders.

Appendix A. Derivation of (2.13)

Lemma 7. *Given that the boundary is sufficiently resolved by the mesh the following inequality holds for $v|_T \in Q_p(T)$ with $T \in \mathcal{T}_\Gamma$,*

$$h^{1/2} \left\| \frac{\partial v}{\partial n} \right\|_{\Gamma \cap T} \leq Cp^2 \|\nabla v\|_T, \quad (2.13)$$

where C is a constant independent of h and p .

Proof. Let $\Gamma_T = T \cap \Gamma$. Note that

$$\left\| \frac{\partial v}{\partial n} \right\|_{\Gamma_T} \leq \|\nabla v\|_{\Gamma_T} \leq \sqrt{\int_{\Gamma_T} \nabla v \cdot \nabla v \, dV} \leq |\Gamma_T|^{1/2} \|\nabla v\|_{L^\infty(T)}. \quad (A.1)$$

Furthermore, using the following inequality from (4.6.1) in Theorem 4.76 in [16]

$$\|v\|_{L^\infty(T)} \leq C \frac{p^2}{|T|^{1/2}} \|v\|_T$$

gives us

$$\|\nabla v\|_{L^\infty(T)}^2 \leq \sum_{i=1}^d \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(T)}^2 \leq C \frac{p^4}{|T|} \sum_{i=1}^d \left\| \frac{\partial v}{\partial x_i} \right\|_T^2 \leq C \frac{p^4}{|T|} \|\nabla v\|_T^2. \quad (A.2)$$

Assuming that the boundary is sufficiently resolved by the mesh there must exist a constant, C , such that

$$|\Gamma_T| \leq Ch^{d-1}. \quad (A.3)$$

Given that our mesh is non-degenerate we also have $|T| \geq Ch^d$. Combining this with (A.1), (A.2) and (A.3) gives us (2.13). \square

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